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Upstream influence

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The phenomenon under investigation occurs in various flow systems characterized by being dispersive to small-amplitude waves, but such that phase and group velocities approach the same finite limit at extreme wavelengths: for example, water in an open channel, density-stratified fluid flowing between horizontal boundaries, and rotating fluid contained in a tube. It is well known in each of these examples that, when a solid body is moved steadily at a subcritical velocity (i.e. less than the long-wave limit) relative to the undisturbed fluid, the body experiences resistance accountable to the continual development of a pattern of waves on the leeward side. But in these circumstances there is a second effect, upstream influence, consisting of a disturbance in the form of a uniform long wave that extends to a steadily increasing distance ahead of the body; and this has for some time been the subject of a controversy, particularly with regard to the question whether or not it is always present. The aim of the paper is to establish the principle that this second effect is inevitably coexistent with the first, being an essential component of the mechanism of wave resistance.

CONTENTS

1. Introduction

The object of this paper is to establish a firm interpretation of the phenomenon commonly called *upstream* or *forward influence*, or in its strongest manifestations *blocking*, about which there has been controversy in recent years. It is associated generally with the phenomenon of wave resistance arising when solid bodies are propelled along straight paths in dispersive systems, and the present account reveals a vital interdependence of the two phenomena. Three specific flow systems are considered having basic attributes in common: open-channel flow (§2), density-stratified flow (treated briefly in §3.5), and rotating flow (§§3, 4). Although in all these examples the phenomenon under study may be significantly complicated by effects of viscosity, it does not depend essentially on them, and in this theoretical discussion only inviscid fluids will be considered.

The water-wave problem examined in §2 serves as a helpful prototype, and in fact it originally suggested to the writer how the more difficult problem of rotating flow could be tackled. To put the general topic in focus, suppose that a solid obstacle is moved at constant horizontal velocity U through water initially at rest, or (what amounts to the same situation observed from another frame of reference), the obstacle is fixed in an initially uniform stream. It is well known that the relative flow becomes steady near the obstacle, a wave-train forms behind it if the velocity U is subcritical (i.e. less than the velocity of extremely long waves), and consequently the obstacle suffers wave resistance. The question of present interest is whether these familiar effects are accompanied by a disturbance of the water in front of the obstacle, extending eventually to great distances ahead. Actually, as will be shown by fairly simple means in §2, water is steadily accumulated in front so that a rise in surface level spreads continually farther forward.

From the standpoint of theoretical model-making the question of upstream influence in open-channel flow is not presented in any urgent sense, and indeed it appears to have been overlooked in most previous treatments of the wave-resistance phenomenon (e.g. Lamb 1932, §§246, 249). The steady state established ahead of the obstacle after passage of the front of the upstream disturbance is qualitatively similar to the original one, differing only in depth and (constant) velocity, and the usual steady-flow theory comprehends the whole range of upstream possibilities without needing to discriminate how they might be realized. However, the question bears with much greater consequence on the theory of wave resistance in rotating and stratified fluids. Here very special models have generally been used, which depend crucially on the hypothesis that the effects of the obstacle do not extend indefinitely far forward. For example, a tractable model of the flow past symmetrical bodies in rotating fluids is provided by the assumption that the flow upstream is steady with uniform axial and angular velocity, for then the full equations of steady motion can be reduced exactly to a *linear* equation for the stream function (see Long 1953; Squire 1956, §§3.4–3.6).

But in the light of the fact that upstream influence exists, this cannot be upheld as an exact model—except by the extremely artificial supposition that the disturbance propagating forward cancels out pre-existing non-uniformities in the distributions of axial and angular velocity.

The problem to be examined in §§3 and 4 concerns the uniform translation of a body along the axis of a rotating fluid. To summarize what is known about flows arising in this way, it is essential to refer to the Rossby number $\mathsf{Ro} = U/\Omega l$, where U is the axial velocity of the body, l its maximum (or otherwise typical) radius and Ω the angular velocity of the undisturbed fluid. The interpretation of events when $\mathsf{Ro} \ll 1$ is soundly established: columns of fluid are then pushed ahead of the body and drawn behind it, in the manner originally demonstrated in experiments by Taylor (1922). [At this point a distinction should be recognized between this strong effect, for which the term *blocking* is appropriate, and the weaker form of columnar disturbance that occurs at larger Ro, in which fluid particles acquire a steady motion but with velocities considerably smaller than U.] The condition $\mathsf{Ro} \ll 1$ justifies linearization of the equations of motion, and on this basis a great deal of theory has been developed concerning strong Taylor columns. The conclusions of such theory are generally in accord with observation at small Ro. If Ro is not particularly small, however, a far more difficult theoretical problem is posed, and almost all work on it so far has rested in some degree on conjecture. [The same statement can be made about work on the analogous problem of stratified flow.] The simplification ensuing from the neglect of upstream influence has already been noted, and this has been exploited in many theoretical studies. The equations become intrinsically non-linear if the hypothesis of steady flow is rejected, but progress has been made by Trustrum (1964) and Stewartson (1968) using linearizing approximations of the Oseen type. Their work has given some strong indications that an upstream effect is never absent from the solution of a properly posed initial-value problem. Again on the basis of linearized theory, Lighthill (1967) has presented a very comprehensive account of the possible wave motions in a rotating fluid, showing how they can comprise forward and rearward columnar disturbances as well as the oscillatory lee-wave system that is observed at moderate Ro. Thus the *possibility* of forward influence at all Ro is already well demonstrated, but not so far its *inevitability*. The existing theory is critically surveyed in chapter 4 of the recent monograph by Greenspan (1968), who concludes tentatively (pp. 222, 224) in favour of the principle that columnar formation is a universal feature of rotating flows.

The available experimental evidence is also conflicting. In experiments with a body having a hemispherical front and conical tail, Long (1953) detected no forward effect for $\mathsf{Ro} > 0{\cdot}23$; and he used the hypothesis of undisturbed upstream flow to obtain a satisfactory theoretical account of the lee-waves observed in his experiments. Recently, however, applying a sensitive new technique to visualize the flow ahead of a sphere, Pritchard (1968) has found evidence of forward influence at Rossby numbers much higher than the limit suggested by Long. Blocking was observed at all $\mathsf{Ro} < 0{\cdot}7$ approximately, and the presence of the more feeble type of columnar disturbance was still indicated at values of Ro as high as about 2.

As far as the writer is aware, there is as yet only one example in which the existence of upstream influence at finite Rossby number has been predicted with certainty. This is the case of an axisymmetric cavity advancing into rotating liquid contained in a long tube, which was treated both experimentally and theoretically by Benjamin & Barnard (1964). By considering a momentum balance they showed that a steady flow relative to the cavity is impossible, and in an appendix to their paper L. E. Fraenkel presented an alternative argument which confirmed this conclusion. The theoretical prediction was borne out by the experiments, which revealed a remarkably vigorous Taylor column at a Rossby number about 0·8 based on the cavity radius.

The present study is comparable with the theoretical work just mentioned, in that the approach is directed towards establishing the necessity of upstream influence indirectly without requiring the non-linear equations of unsteady motion to be solved. In both the water-wave problem and the problem of rotating flow, approximate energy arguments are found to be inherently inconclusive in relation to the question of upstream effects, as also are momentum arguments referring to only part of the overall disturbance created by the obstacle. But in the water-wave problem a definite answer is provided by a reckoning of the total horizontal momentum imparted to the water: curiously, it is only from this particular physical viewpoint that the intrinsicality of upstream disturbances becomes conspicuous. Simple momentum considerations are useless for the problem of rotating flow; and in §3 a quantity termed the axial impulse is defined as an aggregate property of the flow and hence is shown to have a formal significance akin to that of total momentum in the water-wave problem. The impulse principle established in §3 is applied in §4 as the basis of a *reductio ad absurdum*. Thus, in the second problem as in the first, the suppositions that wave resistance is manifested but that upstream influence is absent are shown to be contradictory.

The aim of the following analysis is to settle a point of general principle, and no attempt is made to estimate the actual errors—for instance, in drag calculations—that may be consequent upon the neglect of upstream influence. In the view of Greenspan (1968, p. 215) and others who have favoured the present conclusion, the hypothesis of no upstream disturbance may yet be confirmed to give excellent approximations over a wide range of parameters. The material of this paper is not incompatible with such a view, but it does sharpen the reason for circumspection about calculations that take no account of upstream effects.

2. Upstream influence in open-channel flow

We consider the problem illustrated in figure 1. Water is contained in a horizontal open channel of rectangular cross-section and unit breadth. The depth of the undisturbed water is h_0, so that $c_0 = (gh_0)^{\frac{1}{2}}$ expresses the speed at which small-amplitude waves of extreme length propagate relative to the water. The situation shown in figure 1(*a*) is that a cylindrical obstacle spanning the channel is propelled horizontally at a constant velocity $U < c_0$, disturbing the water which was originally at rest everywhere. It is supposed that the obstacle has been moving for only a limited time, and the figure indicates the finite extent of the developing disturbance. Figure 1 (*b*) is a view of the same situation from a frame of reference moving with the obstacle: the undisturbed water then approaches with velocity U, and also recedes with this velocity beyond the region influenced by the obstacle. The assumed condition $U < c_0$ connotes that the undisturbed stream is *subcritical* [i.e. according to the usual definition, Froude number $F = U/(gh_0)^{\frac{1}{2}} < 1$].

The behaviour of the water in the vicinity of the obstacle can be considered as axiomatic, being well established by observation and previous theory. It is that the relative flow tends to become steady, and a stationary wave-train is formed on the rearward side. Thus, after a sufficient time, a virtually steady flow will be observed between cross-sections such as B' and C' in figure 1 (*b*). The wave drag on the obstacle is a uniquely determinate property of this local flow; but obviously the consideration of possible régimes between B' and C' cannot by itself answer the question whether or not the steady uniform flow at B' is the same as the original one. Our aim is to show that the answer is always negative. There is in fact a continual accumulation of water in front, causing a rise in surface level that eventually covers any distance, however large, measured from the obstacle. This effect is indicated in figure 1, and we shall refer to it as the forward surge.

2.1. *Conservation equations*

Let x denote the horizontal co-ordinate, increasing in the direction forward of the obstacle, and y the vertical co-ordinate measured above the channel bottom. Let u, v denote the respective velocity components, p pressure, ρ density, and $h(x,t)$ the local depth. We define

$$M = \int_0^h \rho u \, dy, \tag{2.1}$$

$$S = \int_0^h (p + \rho u^2) \, dy, \tag{2.2}$$

$$E = \int_0^h \tfrac{1}{2}\rho(u^2+v^2)\, dy + \tfrac{1}{2}\rho g h^2, \tag{2.3}$$

$$W = \int_0^h \{p + \tfrac{1}{2}\rho(u^2+v^2) + \rho g y\} u \, dy. \tag{2.4}$$

Here M is interpretable as the mass flux through a vertical section, and also as the density with respect to x of horizontal momentum. S is conveniently called *flow force*, being the sum of horizontal pressure force and momentum flux. E is energy density, and W is the sum of energy flux and the rate of working by pressure across a vertical section. Accordingly, for all parts of the flow except the stretch occupied by the obstacle, the equations expressing conservation of mass, momentum and energy are respectively

$$\left.\begin{aligned} \rho(\partial h/\partial t) + \partial M/\partial x &= 0, \\ \partial M/\partial t + \partial S/\partial x &= 0, \\ \partial E/\partial t + \partial W/\partial x &= 0. \end{aligned}\right\} \tag{2.5}$$

These equations are exact but too complicated for direct use. To progress we have to consider approximate forms of them applicable to waves whose properties, such as wavelength in oscillatory trains, vary only gradually and whose amplitudes are small. The method is familiar from much recent theoretical work on waves in non-linear dispersive systems, and the most helpful account for present purposes is by Whitham (1962): reference is made to his paper for several results used as follows. It has to be assumed that the variations in wave properties are sufficiently gradual, and the overall length of a train is great enough, for local averages to be definable over distances large compared with wavelength but small compared with the overall length. On the understanding that a fairly

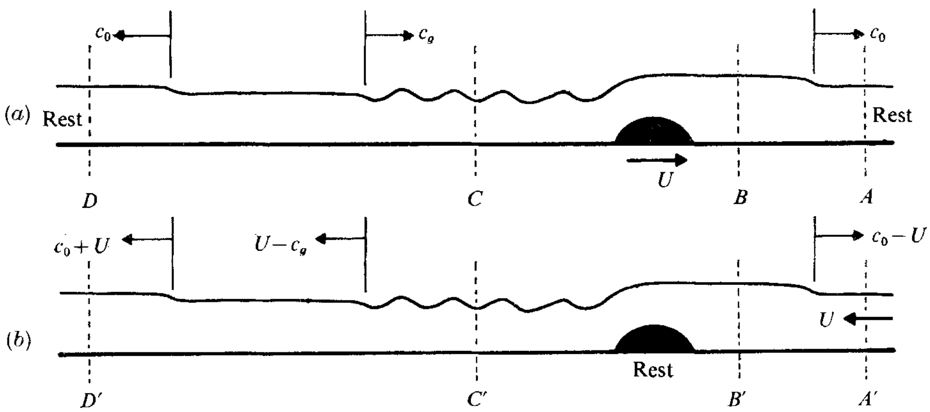

FIGURE 1. Illustration of water-wave problem: (*a*) obstacle propelled at constant velocity U in water originally at rest; (*b*) obstacle fixed in stream approaching with velocity U.

long time has elapsed since the start of the motion, these assumptions appear very reasonable in the situation studied here; and so, as a plausible basis for our argument, we may simply follow Whitham and other writers and take for granted that averaged conservation equations become valid at sufficiently large times. [In anticipation of a later part of this paper, however, it is worth noting that the equations to be considered here may also be justified formally as asymptotic approximations for $t \to \infty$. The procedure in view is one used in §4.3, which defines spatial averages over distances proportional to t^β with $\frac{1}{2} < \beta < 1$. By this means the asymptotic validity of the following second-order approximations may be inferred from known results of linearized surface-wave theory applied to the transient problem (e.g. see Stoker 1953; Maruo 1957). The main facts are, first, that transients in the vicinity of the obstacle decay proportionally to $t^{-\frac{1}{2}}$ and, secondly, that transients at the rear end of the oscillatory train shown in figure 1 are confined to a region whose length is $O(t^{\frac{1}{3}})$, whereas the overall length of the train is $O(t)$. Also, a long wave such as the forward surge shown in figure 1 can have an oscillatory frontal region, but the length of this is only $O(t^{\frac{1}{3}})$. Thus the terminal zones of either type are obliterated by the specified averaging process.]

To begin we need to recall the first approximation to the phase velocity c at which periodic waves propagate relative to the water. This is

$$c = (g\kappa^{-1}\tanh\kappa\bar{h})^{\frac{1}{2}}, \tag{2.6}$$

where $\bar{h}$ is the mean depth and $\kappa = 2\pi/(wavelength)$. In this approximation the waves are sinusoidal, and the error in (2.6) is proportional to the square of their amplitude a. The corresponding group velocity is

$$\begin{aligned} c_g = d(\kappa c)/d\kappa &= \tfrac{1}{2}(1+2\kappa\bar{h}\operatorname{cosech} 2\kappa\bar{h})\,c \\ &= \gamma c, \quad \text{say.} \end{aligned} \tag{2.7}$$

The approximations to (2.1)–(2.4) are averages in the sense that has been explained. The locally averaged depth is $\bar{h}$, as already denoted, and a mean transport velocity $\hat{u}$ may be defined such that $\hat{u}\bar{h}$ is the local flow rate $\bar{M}/\rho$. [Note that $\hat{u}$ can differ from the spatial mean value $\bar{u}$ of the horizontal velocity (cf. Whitham 1962, p. 138).] Each of the other quantities is most usefully expressed as the sum of, first, its value for a uniform flow with depth $\bar{h}$ and velocity $\hat{u}$ and, second, the change arising from the superposition of a wave-train on this flow. Thus, following Whitham (1962, §3), we take

$$\bar{M} = \rho\hat{u}\bar{h}, \tag{2.8}$$

$$\bar{S} = \rho(\hat{u}^2\bar{h} + \tfrac{1}{2}g\bar{h}^2) + (2\gamma - \tfrac{1}{2})\,E_w, \tag{2.9}$$

$$\bar{E} = \rho(\tfrac{1}{2}\hat{u}^2\bar{h} + \tfrac{1}{2}g\bar{h}^2) + E_w, \tag{2.10}$$

$$\bar{W} = \rho\hat{u}(\tfrac{1}{2}\hat{u}^2\bar{h} + g\bar{h}^2) + \{(2\gamma+\tfrac{1}{2})\,\hat{u} + \gamma c\}\,E_w, \tag{2.11}$$

in which $$E_w = \tfrac{1}{2}\rho g a^2.$$

Under the stated assumptions, three equations like (2.5) relating $\bar{h}, \bar{M}, \bar{S}, \bar{E}, \bar{W}$. may be inferred to describe gradually varying properties. It was pointed out by Whitham that in these equations $\bar{h}, \hat{u}, E_w$ and κ (which determines c and γ) can be considered as the basic dependent variables. He also noted that a system of four simultaneous equations is completed by the kinematical condition

$$\frac{\partial\kappa}{\partial t} + \frac{\partial\omega}{\partial x} = 0,$$

in which ω is wave frequency.

The general solution of this system, in the form applicable to small-amplitude wave motions generated from rest, was given by Whitham;† and the solution applicable at present in the fixed frame of reference can be particularized further by the condition that in the vicinity of the obstacle it should be cancelled by the operator $(\partial/\partial t + U\partial/\partial x)$. In this way we ascertain the possibilities illustrated in figure 1 (*a*). These are: (1) The forward surge which is uniform over virtually all its length and whose front advances at the velocity c_0, so drawing steadily

† Note that although the system of averaged equations applies in either frame of reference, the details are considerably simpler in the fixed frame. Both $\hat{u}$ and $\bar{h} - h_0$ are then $O(E_w)$, so that to the first order in E_w four terms in (2.9)–(2.11) are immediately negligible. Also to this order no Doppler effect is included in ω, and an equation for E_w alone can be separated from the system.

ahead of the obstacle moving at the subcritical velocity U. (2) The oscillatory wave-train which is stationary relative to the obstacle so that $c = U$, which is uniform in wave amplitude over most of its length, and whose back end advances at the velocity $c_g = \gamma U < U$, so falling steadily behind the obstacle. (3) The uniform rearward surge whose end recedes at the velocity $-c_0$. Considering these features as established by Whitham's theory, we shall proceed with the interpretation by applying the conservation equations in their form integrated over particular stretches of the channel. By virtue of the fact that the relative flow immediately over the obstacle is steady, the effects of the obstacle can then be included accurately just as if they were concentrated at a point on the x-axis. For instance, the drag is equivalent to a discontinuity in $\overline{S}$.

The values of the mean depth $\overline{h}$ in the forward surge, the oscillatory train and the rearward surge will be denoted respectively by

$$h_0(1+\delta_+), \quad h_0(1+\delta_w), \quad h_0(1+\delta_-).$$

The respective transport velocities, which like the above δ's are all $O(E_w)$, will be denoted by $\hat{u}_+, \hat{u}_w, \hat{u}_-$.

2.2. *The classical argument*

This establishes the relationship between wave resistance $\mathscr{D}$ and amplitude a by considering the energy balance for a region such as between A and C in figure 1 (a) [see Lamb 1932, § 249]. The fact needing emphasis here is that this approach oddly evades the question of forward influence: that is to say, a definite expression for $\mathscr{D}$ is obtained without the need for any estimate of the flow ahead of the obstacle.

At the vertical section C through the wave-train, the first approximation to $\overline{W}$ given by (2.11) is

$$\overline{W}_c = \rho g h_0^2 \hat{u}_w + \gamma U E_w, \tag{2.12}$$

and $\overline{W} = 0$ at A. The integrated equation of energy conservation hence confirms what is directly obvious, that the rate of increase of the total energy between A and C equals the sum of (2.12) and the propulsive power $\mathscr{D}U$—which is equivalent to a discontinuity in $\overline{W}$. Thus, taking the first approximation to $\overline{E}$ given by (2.10), we deduce

$$\mathscr{D}U = \rho g h_0^2\{(c_0 - U)\,\delta_+ + U\delta_w - \hat{u}_w\} + (1-\gamma)\,E_w. \tag{2.13}$$

But, to the same approximation, the condition of mass conservation for the region between A and C is

$$h_0\{(c_0 - U)\,\delta_+ + U\delta_w\} = h_0\hat{u}_w, \tag{2.14}$$

showing that the first group of terms on the right-hand side of (2.13) cancels out. We are left with

$$\mathscr{D} = (1-\gamma)\,E_w = \tfrac{1}{4}\rho g a^2(1 - 2\kappa h_0 \operatorname{cosech} 2\kappa h_0), \tag{2.15}$$

which recovers the classical result.

Nothing different is learnt from energy considerations for the whole region disturbed by the obstacle [i.e. between A and D in figure 1 (a)]. The rate of energy increase is then $(U-c_g)E_w = U(1-\gamma)E_w$ due to the oscillatory wave-train, and the contributions from changes in mean surface level again cancel out in consequence of mass conservation. [The result (2.15) can also be derived rather simply by a momentum argument referring to the steady flow observed near the obstacle in the moving frame (see Whitham 1962, §5). One then has $\mathscr{D} = S_+ - S_w$, where S_+ and S_w are the respective flow-force values at sections such as B' and C' in figure 1 (*b*).]

Some further information is gained from a momentum balance for the classical model. At C the first approximation to $\bar{S}$ given by (2.9) is

$$\bar{S}_C = \tfrac{1}{2}\rho g h_0^2 + \rho g h_0^2 \delta_w + (2\gamma - \tfrac{1}{2}) E_w, \tag{2.16}$$

in which the term $\frac{1}{2}\rho g h_0^2$ is just S_A, the pressure force in the undisturbed water. According to the integrated equation of momentum conservation, the sum of $\bar{S}_C - S_A$ and the drag on the obstacle equals the rate of increase of horizontal momentum between A and C. Hence, using (2.15) to express $\mathscr{D}$, we obtain

$$\rho g h_0^2 \delta_w + (\tfrac{1}{2}+\gamma) E_w = \rho h_0 \{(c_0 - U)\hat{u}_+ + U\hat{u}_w\}. \tag{2.17}$$

But, to the present order, mass conservation obviously requires

$$\hat{u}_+ = c_0 \delta_+ \tag{2.18}$$

as well as (2.14), and by means of these two equations $\hat{u}_+$ and $\hat{u}_w$ can be eliminated from (2.17). There follows after some reduction

$$\delta_+ - \delta_w = \frac{(\frac{1}{2}+\gamma)E_w}{\rho g h_0^2 (1-F^2)} = \frac{(1+2\gamma)\Delta}{4(1-F^2)}, \tag{2.19}$$

where $F = U/c_0 < 1$ and $\Delta = (a/h_0)^2$. [The same result is obtainable from an energy balance in the moving frame, for a region such as between A' and C' in figure 1 (*b*).]

Equation (2.19) shows that necessarily $\delta_+ > \delta_w$. That is, the mean surface level behind the obstacle is always lower than that in front. But we still have no reason to exclude $\delta_+ = 0$ as a possibility.

2.3. *Necessity of the forward surge*

The preceding subsection has exemplified an idea which, though negative, seems vital to the understanding of our general subject, and which is to be met again later in the paper. This is the curious evasiveness of evidence about upstream influence when wave-resistance phenomena are examined by means of energy arguments, or momentum arguments referring to only part of the overall wave system. However, the rôle of forward disturbances becomes conspicuous in another approach, which refers to the net momentum (or, more generally, net impulse—see §3 below) of the entire system. On this basis the necessity of upstream influence in rotating flows will be demonstrated in §4 by a *reductio ad absurdum*, and we note here the particularly simple counterpart of this argument for the water-wave problem.

Applied in the fixed reference frame to a stretch of the channel including all water disturbed by the obstacle, as between A and D in figure 1 (*a*), the momentum equation becomes

$$\mathscr{D} = \frac{d\mathscr{M}}{dt}, \tag{2.20}$$

where $\mathscr{M}$ is the total momentum imparted to the water in the direction of travel of the obstacle. Now suppose there is no forward effect (i.e. $\delta_+ = 0$). According to (2.19), δ_w is then negative and hence the mass flux in the wave-train must be negative. Mass conservation further requires that δ_+ is positive, in order to accommodate the backward flow in the wave-train. Hence, as the rearward surge has to recede into the still water at D, the mass flux in it also must be negative. Thus the total momentum is always negative, which contradicts (2.20), and so the assumption of no forward effect is shown to be incorrect.

2.4. *Completion of the solution*

The argument just presented can be extended to give explicit estimates of δ_+, δ_w and δ_-. First, (2.20) is expressed in its detailed form

$$\mathscr{D} = \rho h_0\{(c_0 - U)\,\hat{u}_+ + (U - c_g)\,\hat{u}_w + (c_0 + c_g)\,\hat{u}_-\}, \tag{2.21}$$

which, after substitution of (2.15), then (2.14), (2.18) and the further relation $\hat{u}_- = -c_0\delta_-$ obviously required by mass conservation, can be rearranged to give

$$(1-F)(1+F-\gamma F)\,\delta_+ + (1-\gamma)\,F^2\delta_w - (1+\gamma F)\,\delta_- = \tfrac{1}{2}(1-\gamma)\,\Delta. \tag{2.22}$$

Secondly, the overall condition of mass conservation is expressed by

$$(c_0 - U)\,\delta_+ + (U - c_g)\,\delta_w + (c_0 + c_g)\,\delta_- = 0,$$

i.e.

$$(1-F)\,\delta_+ + (1-\gamma)\,F\delta_w + (1+\gamma F)\,\delta_- = 0. \tag{2.23}$$

Solving the set of simultaneous equations (2.19), (2.22) and (2.23) we obtain finally

$$\left.\begin{aligned} \frac{\delta_+}{\Delta} &= \frac{(1-\gamma)(2-F+2\gamma F)}{8(1-F)(1-\gamma F)}, \\ \frac{\delta_w}{\Delta} &= -\frac{6\gamma + F(1-\gamma)(2\gamma-1)}{8(1+F)(1-\gamma F)}, \\ \frac{\delta_-}{\Delta} &= -\frac{(1-\gamma)(2+F-2\gamma F)}{8(1+F)(1+\gamma F)}. \end{aligned}\right\} \tag{2.24}$$

We also have, by rearranging (2.6),

$$F^2 = \frac{\tanh \kappa h_0}{\kappa h_0},$$

from which κh_0 can be calculated as a function of F, and then γ can be found from (2.7).

The results (2.24) are plotted as functions of F in figure 2. It is of interest that δ_+ is positive and δ_w, δ_- both negative, because this implies that $\hat{u}_+$, $\hat{u}_w$, $\hat{u}_-$ are all positive.† Thus each of the three parts of the wave system adds towards the

† Note that in the forward and rearward surges the transport velocities $\hat{u}_+$, $\hat{u}_-$ are the same as the respective mean horizontal velocities, but in the oscillatory wave-train the mean velocity is given by

$$\bar{u}_w = \hat{u}_w - E_w/\rho U h_0$$

(cf. Whitham 1962, p. 139). The substitution of this expression in (2.21) shows that $\bar{u}_w$ is negative (i.e. the spatially averaged velocity in the wave-train is directed to the rear, even though the mass transport is forward), and also that the average of the horizontal velocity over the whole wave system is zero.

total momentum $\mathcal{M}$. For finite Δ, both δ_+ and δ_w appear to be unbounded in the limit $F \uparrow 1$ ($\gamma \uparrow 1$). No significance should be attached to the results with F close to 1, however, since the approximations used for the oscillatory-wave properties are then unreliable.

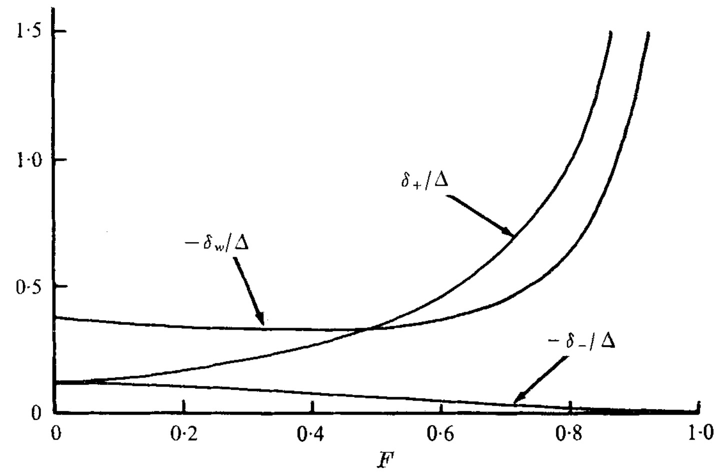

FIGURE 2. Mean values δ_+, $-\delta_w$, $-\delta_-$ of surface displacements in forward surge, oscillatory wave-train and rearward surge, plotted as functions of Froude number F.

The case of very deep water corresponds to the limit $F \downarrow 0$ ($\gamma \downarrow \frac{1}{2}$). The asymptotic relationships given by (2.24) are

$$(\delta_+, \delta_w, \delta_-) \sim (\tfrac{1}{8}, -\tfrac{3}{8}, -\tfrac{1}{8})\Delta,$$

and so the actual changes in surface level are proportional to a^2/h_0, thus becoming insignificant when h_0 is sufficiently large. It is perhaps worth emphasis that this extreme case is *not* analogous to the case of an unbounded rotating fluid. When a body is propelled horizontally near the surface of infinitely deep water, no forward surge can be generated because there is no long-wave mode having finite momentum or energy. But a continuous spectrum of such modes exists for an infinite rotating fluid (e.g. see Lighthill 1967).

3. The impulse principle

We turn to the problem of a rigid body moved along the axis of rotation in an incompressible rotating fluid. It is supposed that the fluid extends to infinity in the axial direction upstream and downstream, but has a rigid cylindrical boundary of finite diameter. This case is in practical respects more significant than the case of an unbounded fluid, which can be treated by an extension of the analysis developed here but which presents additional complications; so attention is restricted to the first case for the sake of simplicity. The inevitability of upstream influence will be demonstrated later, in §4, by an argument closely resembling the preceding one. In contrast with the water-wave problem, however, a simple reckoning of total momentum is now unavailing, as is shown by the following considerations.

3.1. *Implications of momentum conservation*

The problem is illustrated in figure 3, whose two parts are views respectively from a fixed frame of reference and from one moving with the body. We assume that the axial velocity U of the body in figure 3(*a*) [and correspondingly the axial velocity of the undisturbed fluid in figure 3(*b*)] is constant, and that the body experiences resistance consequent upon the formation of lee waves. Let (x, r, θ) denote cylindrical polar co-ordinates, with x increasing in the forward

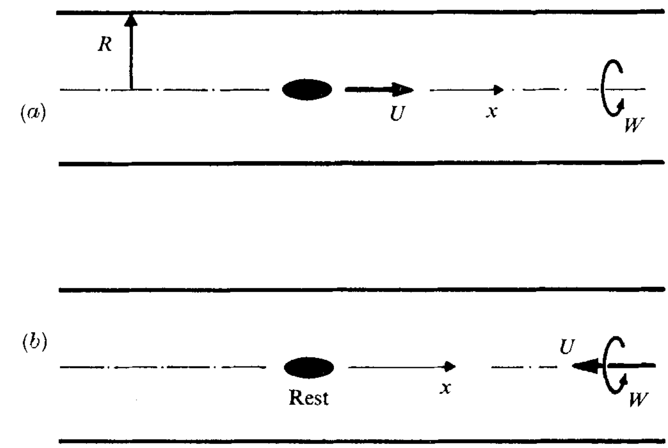

FIGURE 3. Illustration of rotating-fluid problem: (*a*) body propelled with axial velocity U in fluid that originally has azimuthal velocity $W(r)$ and no axial motion; (*b*) body fixed in swirling flow approaching with axial velocity U.

direction along the axis of rotation, and let (u, v, w) denote the respective velocity components. Let R denote the radius of the boundary. Accordingly, quantities corresponding to (2.1) and (2.2) are defined by

$$M = \rho \int_0^{2\pi} \int_0^R ur\,dr\,d\theta, \tag{3.1}$$

$$S = \int_0^{2\pi} \int_0^R (p + \rho u^2)\, r\,dr\,d\theta. \tag{3.2}$$

The equation expressing conservation of momentum is as before

$$\frac{\partial M}{\partial t} + \frac{\partial S}{\partial x} = 0. \tag{3.3}$$

But now, since obviously the body cannot displace the fluid at $x = \pm\infty$ when only a finite time has elapsed from the start of the motion, conservation of mass requires M to be constant. Specifically, $M = 0$ in the fixed reference frame, and $M = -\pi R^2 \rho U$ in the moving frame. Hence (3.3) shows that the value of flow force S must be constant everywhere in front of the body and also constant everywhere behind: that is, the value S_+ immediately in front is the same as at $x = \infty$, and the value S_w in the lee-wave system immediately behind is the same as at $x = -\infty$. Since $S_+ - S_w$ equals the drag $\mathscr{D}$ on the body, this means that the propulsive force is exactly balanced by a reaction in the fluid at infinity. Denoting pressure perturbations by p' (which are, of course, constant over cross-sections where the fluid is undisturbed), we have therefore

$$\mathscr{D} = \pi R^2[p'_{x=\infty} - p'_{x=-\infty}]. \tag{3.4}$$

Clearly it is only the difference in pressure that matters: for example, if the pressure level at $x = \infty$ were fixed, then the drag on the body would cause a suction $\mathscr{D}/\pi R^2$ at $x = -\infty$.

Thus we see that momentum considerations can tell us nothing about the flow generated by the body. Observed in the fixed frame the fluid does have a definite amount of momentum, which is easily found to be $\rho U \times$ (*volume of body*) in the direction *opposite* to the direction of travel of the body [see appendix C, equation (C 9)]; but this is constant and therefore immaterial to the mechanism of wave resistance. Another basic physical difference between this and the water-wave problem appears from energy considerations. In the fixed frame the agency propelling the body does work at the rate $\mathscr{D}U$, and as before the supplied energy is manifested in the wave system. But in the moving frame, where the body appears stationary and the force applied to it does no work, the pressure difference between the undisturbed flow far upstream and far downstream amounts to an energy supply at the rate $\mathscr{D}U$ (i.e. pumping work must be done to maintain the flow past the intervening obstruction). In this way the energy of the lee waves is still supplied externally, not as in the water-wave problem at the expense of the kinetic energy of the original stream.

3.2. *Derivation of relationship between impulse and drag*

To complete a line of reasoning similar to that followed in §2, a quantity P_x termed the axial impulse of the fluid needs to be considered. As primarily defined P_x is an aggregate property of the flow caused by the body, but we shall find that $dP_x/dt = \mathscr{D}$. Thus P_x has the same rôle as total momentum in the water-wave problem, although it must mean something different in physical terms. For the moment, however, we defer matters of physical interpretation and proceed formally.

The following derivation deals with the flow as observed in the moving frame of reference, at times subsequent to the initial acceleration of the body up to its steady axial velocity U. That is, the body is taken to be at rest, the velocity vector $\mathbf{u}$ is therefore tangential along the fluid surface $\mathscr{S}$ in contact with the body, and

$$\mathbf{u} = (u, v, w) \to [-U, O, W(r)] \quad \text{for} \quad x \to \pm\infty.$$

We also consider a hypothetical surface Σ that includes the rigid cylindrical boundary $r = R$ and is closed by cross-sections at infinity upstream and downstream. The fluid is unperturbed from the primary state of motion $\mathbf{u} = (-U, 0, W)$ on each of the latter parts of Σ, and with regard to the cylindrical part we rely on the fact that $\mathbf{u}$ and the vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ are necessarily tangential along it. The equation of motion is considered in the form (cf. Batchelor 1967, p. 382)

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega} - \nabla\left(\frac{p}{\rho}+\tfrac{1}{2}|\mathbf{u}|^2\right), \tag{3.5}$$

from which Helmholtz's vorticity equation may be obtained by taking the curl of both sides, thus

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}). \tag{3.6}$$

We define

$$\left.\begin{aligned} \mathbf{P} &= \mathbf{P}_V + \mathbf{P}_{\mathscr{S}}, \\ \mathbf{P}_V = \tfrac{1}{2}\rho\int_V \mathbf{x}\times\boldsymbol{\omega}\,d\tau, &\quad \mathbf{P}_{\mathscr{S}} = \tfrac{1}{2}\rho\int_{\mathscr{S}} \mathbf{x}\times(\mathbf{u}\times d\mathbf{s}),\end{aligned}\right\} \tag{3.7}$$

where $\mathbf{x}$ is the Euclidean position vector and $d\tau$ the element of volume.† The volume V is bounded internally by $\mathscr{S}$ and externally by Σ as just prescribed. Here and later, when surface integrals over Σ will be considered, the surface element $d\mathbf{s}$ is understood as a vector in the normal direction away from the interior of V. Since in the undisturbed fluid $\mathbf{u}$ and $\boldsymbol{\omega}$ are everywhere parallel to the axis of rotation, we see from (3.7) that the axial component P_x of $\mathbf{P}$, in which our main interest lies, is due wholly to the disturbance caused by the body. In the case of axial symmetry, for instance, $(P_V)_x$ arises only from the azimuthal component of vorticity.

Since V is a fixed volume, we have

$$\begin{aligned} \frac{\partial \mathbf{P}_V}{\partial t} &= \tfrac{1}{2}\rho\int_V \mathbf{x}\times\frac{\partial \boldsymbol{\omega}}{dt}\,d\tau \\ &= \tfrac{1}{2}\rho\int_V \mathbf{x}\times\{\nabla\times(\mathbf{u}\times\boldsymbol{\omega})\}\,d\tau,\end{aligned}$$

by virtue of (3.6). This volume integral may be reduced by a formula derived in appendix A, giving

$$\frac{d\mathbf{P}_V}{dt} = \rho\int_V \mathbf{u}\times\boldsymbol{\omega}\,d\tau - \tfrac{1}{2}\rho\int_{\mathscr{S}+\Sigma} \mathbf{x}\times\{(\mathbf{u}\times\boldsymbol{\omega})\times d\mathbf{s}\}. \tag{3.8}$$

No contribution to the surface integral is made from the cylindrical part of Σ, along which $\mathbf{u}$ and $\boldsymbol{\omega}$ are tangential and therefore $\mathbf{u}\times\boldsymbol{\omega}$ is normal, and there is a

† An expression like $\mathbf{P}_V$ has been considered by Lamb (1932, §152) and Batchelor (1967, p. 520) as defining the impulse of a vortex system in an infinite fluid without internal boundaries. The analysis that follows in this subsection is comparable with Batchelor's treatment.

Note that the contribution $\mathbf{P}_{\mathscr{S}}$ to $\mathbf{P}$ could be absorbed into $\mathbf{P}_V$ if $\mathscr{S}$ were taken as the surface of the body, rather than that of the contiguous fluid, and if the volume integral were interpreted as a Stieltjes integral with respect to the discontinuity in velocity between the body and the fluid. Alternatively, it could be postulated that the velocity of slipping past the body is attained continuously through a boundary layer of extremely small thickness, in which case Stokes's theorem gives $\mathbf{P}_{\mathscr{S}}$ as the contribution to $\mathbf{P}$ due to vorticity in the layer.

cancellation of the contributions from the cross-sections at $x = \infty$ and $x = -\infty$. Hence from (3.7) and (3.8) we have

$$\frac{d\mathbf{P}}{dt} = \rho\int_V \mathbf{u}\times\boldsymbol{\omega}\,d\tau + \tfrac{1}{2}\rho\int_{\mathscr{S}} \mathbf{x}\times\left\{\left(\frac{\partial\mathbf{u}}{\partial t}-\mathbf{u}\times\boldsymbol{\omega}\right)\times d\mathbf{s}\right\}$$

$$= \rho\int_V \mathbf{u}\times\boldsymbol{\omega}\,d\tau - \frac{1}{2}\int_{\mathscr{S}} \mathbf{x}\times\{\nabla(p+\tfrac{1}{2}\rho|\mathbf{u}|^2)\times d\mathbf{s}\}, \tag{3.9}$$

by virtue of the equation of motion (3.5).

The surface integral in (3.9) may be reduced by a formula derived in appendix B, giving

$$\frac{1}{2}\int_{\mathscr{S}} \mathbf{x}\times\{\nabla(p+\tfrac{1}{2}\rho|\mathbf{u}|^2)\times d\mathbf{s}\} = \int_{\mathscr{S}} (p+\tfrac{1}{2}\rho|\mathbf{u}|^2)\,d\mathbf{s}. \tag{3.10}$$

And for the volume integral in (3.9) we deduce, using the condition of incompressibility $\nabla\cdot\mathbf{u} = 0$,

$$\int_V \mathbf{u}\times\boldsymbol{\omega}\,d\tau = \int_V \left\{\tfrac{1}{2}\nabla|\mathbf{u}|^2 - \frac{\partial(u_j\mathbf{u})}{\partial x_j}\right\}d\tau$$

$$= \int_{\mathscr{S}+\Sigma} \tfrac{1}{2}|\mathbf{u}|^2 d\mathbf{s} - \int_{\mathscr{S}+\Sigma} \mathbf{u}(\mathbf{u}\cdot d\mathbf{s}). \tag{3.11}$$

The second of these two integrals obviously vanishes; and the contribution to the first integral from Σ is seen to be a vector perpendicular to the axis, being thus immaterial to the axial component of $d\mathbf{P}/dt$. Hence, combining (3.9)–(3.11) and using the notation $P_x = (\nabla x)\cdot\mathbf{P}$, we obtain finally

$$\frac{dP_x}{dt} = -\int_{\mathscr{S}} p\{(\nabla x)\cdot d\mathbf{s}\}. \tag{3.12}$$

This evidently expresses the axial component of the reaction to the hydrodynamic forces on the body, so that

$$\frac{dP_x}{dt} = \mathscr{D}, \tag{3.13}$$

where $\mathscr{D}$ is, as previously implied by this symbol, the external force that must be applied to the body in order to balance hydrodynamic resistance.

The same conclusion holds with respect to the motion observed in the fixed frame of reference. An alternative derivation relating to this case is presented in appendix C, and there the principle expressed by (3.7) and (3.13) is generalized by allowing the axial velocity of the body to be an arbitrary function of time. Thus the effect of accelerating the body from rest may be included in $\mathbf{P}$, although we have no need to bring this complementary aspect into our discussion of upstream influence.

3.3. *Physical interpretation*

The concept of fluid impulse was originally explained by Kelvin in relation to accelerated bodies in infinite fluids (Lamb 1932, §119). In keeping with his definition, $-\mathbf{P}$ may be interpreted as the impulse that, applied instantaneously and suitably distributed through the fluid, would annul the wave motion caused by the presence of the body. More precisely, at a time t_1, say, the developments since an earlier time t_0 could be annulled by the application of an external impulse $-[\mathbf{P}(t_1)-\mathbf{P}(t_0)]$. From the discussion in §3.1 it appears that P_x is identifiable with the time-integral of the reaction in the fluid at infinity, which accords with the present interpretation, and this equivalence will be demonstrated directly in appendix C. In general, when the motion is viewed from a fixed frame of reference and the effect of accelerating the body from rest is included, P_x is the sum of the integrated reaction at infinity and the total momentum imparted to the fluid. We note incidentally that, as Kelvin showed, this interpretation also applies unambiguously to an unbounded fluid, even though the two factors are separately indeterminate in this case (cf. Birkoff 1950, chapter 5, §5; Benjamin & Ellis 1966).

It deserves particular emphasis that the two components of $\mathbf{P}$ defined by (3.7) have quite different significance at large times. There is good reason for assuming, as in the water-wave problem, that the flow near the body becomes virtually steady after a sufficient time, in which event $\mathbf{P}_{\mathscr{S}}$ becomes constant and so has no part in accounting for the drag as given by (3.13). But the drag also becomes constant, being an attribute of the local flow, and therefore $(P_V)_x$ ultimately increases proportionally to time. Even if the assumption of asymptotically steady flow near the body is relinquished, it still appears that $(P_V)_x$ increases without bound when $\mathscr{D}$ has a positive mean value; for (3.13) shows the sum P_x to have this property, whereas the component $(P_V)_x$ evidently cannot have it if the flow velocity past the body remains finite.

The asymptotic property $(P_V)_x \sim \mathscr{D}t$ as $t \to \infty$ is to be the basis of the proof by contradiction in §4: that is, it will be shown incompatible with the supposition that there is no upstream influence. Intuitively the property is nicely in accord with the notion of wave resistance in rotating fluids; for we know that vorticity perturbations spread away from the body as 'inertial waves', and the growth of the volume integral $(P_V)_x$ simply reflects the continual expansion of the zone under the influence of the applied force $\mathscr{D}$.

3.4. *Axisymmetric flows*

We now particularize the components of impulse (3.7) in the forms required for §4, but in passing note two aspects of the more general principle expressed by (3.7) and (3.13). First, the primary rotation need not be uniform: that is, the undisturbed azimuthal velocity W is an arbitrary function of radius. However, the discussion in §4 will be confined to the special case $W = -\Omega r$. Secondly, axial symmetry of the body is unnecessary, but for simplicity it will be assumed in §4.

Under this assumption the axial component of $\mathbf{P}_{\mathscr{S}}$ may be expressed by

$$(P_{\mathscr{S}})_x = \pi\rho \oint_L \tilde{u} r^2 dl, \tag{3.14}$$

where L is the meridional contour of the body extending from its foremost to its rearmost point, dl is the element of arc length, and $\tilde{u}$ is the meridional velocity tangential to L. Furthermore, only the azimuthal vorticity component $\zeta = \partial v/\partial x - \partial u/\partial r$ is relevant to the axial component of $\mathbf{P}_V$: thus

$$(P_V)_x = \tfrac{1}{2}\rho\int_V r\zeta\, d\tau = \pi\rho\int_V \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right) r^2\, dr\, dx. \tag{3.15}$$

3.5. *Application to heterogeneous fluids*

Internal wave motions in heavy density-stratified fluids bounded below and above by fixed horizontal planes present the same difficulty that was explained in § 3.1: namely, wave resistance cannot be related simply to momentum production. This is also true for stratified fluids with only one or with no horizontal boundary, and so it appears that the present principle, of impulse related to the production of vorticity orthogonal to the main direction of motion, is equally vital in this other context.

For two-dimensional motions in a vertical plane (x, y) vorticity may be considered as a vector perpendicular to the plane, and a revised definition of impulse is needed (see Lamb 1932, § 157) if one is to avoid being concerned with the effects of vortex sheets on the vertical side walls that are implicit in the theoretical model. Also, for a stratified fluid, the vorticity equation (3.6) has a further term expressing the rate of vorticity production that is associated with flows through density gradients. If the Boussinesq approximation is allowed (i.e. the inertial effects of density variations are ignored), then the equation of motion is

$$\bar{\rho}\frac{D\mathbf{u}}{Dt} = -\nabla p - g\rho\nabla y, \tag{3.16}$$

where $\bar{\rho}$ is the average density and ∇y represents a unit vector in the vertical direction. Hence, if the fluid is incompressible so that a stream function ψ exists, it follows that the equation for the vorticity $\zeta = -\nabla^2\psi$ is

$$\bar{\rho}\frac{D\zeta}{Dt} = -g\frac{\partial \rho}{\partial x}. \tag{3.17}$$

An area A may be defined with the same significance as the volume V in § 3.2, being bounded internally by the contour $\mathscr{C}$ of the solid obstacle in question. Then, proceeding essentially as before, one obtains

$$\mathscr{D} = \bar{\rho}\frac{d}{dt}\left\{\int_A \zeta y\, dx\, dy + \oint_{\mathscr{C}} \tilde{u} y\, dl\right\}, \tag{3.18}$$

where $\tilde{u}$ is the tangential velocity along $\mathscr{C}$ and the contour integral is taken in the anticlockwise sense if the x-axis is to the right.

On the basis of this result the necessity of upstream influence as a concomitant of internal wave resistance can be demonstrated by the same sort of argument as in § 4 below.

3.6. *Postscript*

Confidence in present ideas may perhaps be reinforced by noting a sidelight on the famous Kutta–Joukowski formula for the lift force on a two-dimensional aerofoil. In exactly the same way that (3.18) is derived, one obtains for the lift on a body in an infinite fluid of uniform density the expression

$$\mathscr{L} = \rho \frac{d}{dt}\left\{\int_A \zeta x\, dx\, dy + \oint_{\mathscr{C}} \tilde{u} x\, dl\right\}. \tag{3.19}$$

Now consider axes moving with an aerofoil that has had a constant velocity U for some time, so that the adjacent flow has become steady relative to these axes and therefore the contour integral in (3.19) is constant. The only contribution to the first integral is from the starting vortex, or sequence of such vortices, which appears to recede downstream with velocity U. According to a very well-known argument, the area integral of the concentrated vorticity being swept away is $-\Gamma$, where Γ is the circulation around the aerofoil, and the distance x entailed in the first integral increases at the rate $-U$. Hence we may conclude that $\mathscr{L} = \rho U \Gamma$, as expected.

The same conclusion may be reached by taking axes fixed relative to the undisturbed fluid. From this viewpoint the area integral in (3.19) becomes constant at sufficiently large times, and we then have

$$\mathscr{L} = \rho \frac{d}{dt} \oint_{\mathscr{C}} \tilde{u} x\, dl = \rho U \oint_{\mathscr{C}} \tilde{u}\, dl = \rho U \Gamma.$$

4. Necessity of upstream influence in a rotating flow

The problem illustrated in figure 3 will now be treated in detail. The undisturbed fluid is supposed to have a constant angular velocity $-\Omega$, and we are concerned with the flow caused by the steady translation of an axisymmetric body along the axis of rotation. It is assumed that a system of lee waves is formed and consequently the body suffers drag. On the further assumption that upstream influence is absent, a description of the complete flow field is obtained, justifiable as an asymptotic approximation for $t \to \infty$. Deductions on this basis are then juxtaposed with the impulse principle and a contradiction is reached, demonstrating that the hypothesis of no upstream influence is incorrect.

Although mathematical rigour is not attempted, the argument depends on conjecture only in the following readily defensible aspects. First, it is taken for granted that the flow near the body becomes steady asymptotically. This supposition appears extremely reasonable, being supported by experimental observation and also by solutions of the linearized initial-value problem (Trustrum 1964; Miles 1969). In §4.1 the exact theory of steady flow arising without upstream influence is outlined, and the lee-wave system is shown to be representable as a superposition of uniform wave-trains. The second aspect in which the argument relies on a plausible conjecture concerns the 'transient' zones that terminate these wave-trains downstream: the supposition is simply that the lengths of these zones are $o(t)$, so that at sufficiently large times they become insignificant compared with the overall lengths of the respective wave-trains [which are necessarily $O(t)$]. As in the water-wave problem, this supposition is supported by the results of linearized dispersion theory, which shows the lengths in question to be

$O(t^{\frac{1}{2}})$. The validity of the approximate description thus inferred for the complete, time-dependent flow is corroborated in §4.2, where, comparably with §2.2 for the water-wave problem, an energy balance is shown to be satisfied. The contradiction of the impulse principle is established in §4.3, and finally, in §4.4, it is confirmed that the contradiction is resolved by allowing for upstream influence.

4.1. *Equations of steady flow*

The more convenient choice here is to take the frame of reference in which the body is stationary [figure 3(*b*)] and the velocity of the undisturbed fluid is $(-U, 0, -\Omega r)$. It is well known that on the assumption of a *steady* flow which is asymptotic to this uniform state upstream (that is, in the absence of upstream influence as we understand it), the full equations of motion are reducible without approximation to a linear equation for the stream function (see Squire 1956, §3.4; or Batchelor 1967, §7.5). Specifically, if the stream function is expressed as the sum $-\frac{1}{2}Ur^2 + \psi(x, r)$, then the perturbation ψ is found to satisfy

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + k^2 \psi = 0, \tag{4.1}$$

in which

$$k = 2\Omega/U. \tag{4.2}$$

The corresponding expression for the velocity field is

$$(u, v, w) = (-U,\ 0,\ -\Omega r) + \frac{1}{r}\left(\frac{\partial \psi}{\partial r}, -\frac{\partial \psi}{\partial x}, k\psi\right), \tag{4.3}$$

where the azimuthal component w is determined by the condition that the circulation $2\pi r w$ is invariant along each stream-surface.

For the flow inside the rigid cylinder, the boundary conditions applying upstream and downstream of the body are

$$\psi(x, 0) = 0, \quad \psi(x, R) = 0. \tag{4.4}$$

A set of fundamental solutions of (4.1) satisfying these conditions is

$$\left.\begin{aligned}\psi_n &= -\frac{2^{\frac{1}{2}} r J_1(j_n r/R)}{R J_0(j_n)} \operatorname{Re}\{\exp(\pm i\kappa_n x + \nu_n)\} \\ &= \phi_n(r) \operatorname{Re}\{\exp(\pm i\kappa_n x + \nu_n)\}, \quad \text{say,}\end{aligned}\right\} \tag{4.5}$$

where the j_n $(n = 1, 2, 3, \ldots)$ are positive zeros of the Bessel function J_1, the ν_n are arbitrary, and

$$\kappa_n^2 = k^2 - (j_n/R)^2 \tag{4.6}$$

(cf. Squire 1956, §3.6). In the derivation of (4.5), the $\phi_n(r)$ are presented as eigenfunctions of the (singular) Sturm–Liouville system

$$\left.\begin{aligned}\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi_n}{dr}\right) + \left(\frac{k^2 - \kappa_n^2}{r}\right)\phi_n = 0,\\ \phi_n(0) = 0, \quad \phi_n(R) = 0;\end{aligned}\right\} \tag{4.7}$$

and the general Sturm–Liouville theory tells us that the eigenvalues $(k^2 - \kappa_n^2)$ are all real and none is negative. The least eigenvalue is $(j_1/R)^2 = (3{\cdot}832/R)^2$, and thus it is seen that a periodic solution is possible [i.e. $\kappa_n \neq 0$ and real in (4.5)] if and only if

$$\left.\begin{aligned} kR \equiv 2\Omega R/U &> 3{\cdot}832, \\ \text{i.e.} \qquad U &< 0{\cdot}522\Omega R. \end{aligned}\right\} \tag{4.8}$$

This condition corresponds to the condition of subcritical flow in the water-wave problem, and it is henceforth *assumed*.

The set (4.5) includes precisely m independent periodic solutions if $j_{m+1} \geqslant kR > j_m$. It appears, therefore, that the lee-wave system supposed to be formed at a sufficient distance behind the body (i.e. far enough so that those components of the full solution having real-exponential dependence on x have decreased to insignificance) is generally representable by the series

$$\psi = \sum_{n=1}^{m} A_n \psi_n = \sum_{n=1}^{m} A_n \phi_n(r) \cos(\kappa_n x + \nu_n), \tag{4.9}$$

where m is determined by the preceding inequalities.

We note that the infinite sequence $\{\phi_n\}$ is orthonormal on $[0, R]$ with respect to the weight-function r^{-1}, thus

$$\int_0^R \phi_p(r)\,\phi_q(r)\frac{dr}{r} = \delta_{pq}. \tag{4.10}$$

The sequence is also complete in L_2, from which fact the generality of the present account of the lee-wave system could be rigorously established.

4.2. *Drag related to wave energy*

It will next be shown how the theoretical model just outlined appears to be consistent, to a first approximation, with an energy balance for the wave system as it evolves with time. Thus again, as in the water-wave problem, we appreciate that approximate energy considerations tend to reveal nothing contrary to the hypothesis of uniform steady flow upstream. Emphasis of this aspect seems essential to the defence of present views about upstream influence.

First, the drag $\mathscr{D}$ on the body is expressed by evaluating the reduction in flow force S [as defined by (3.2)] between the steady flows upstream and downstream. Using the fact that stagnation pressure is a function only of the total stream function, one readily derives [cf. Benjamin 1962, equation (A 22)]

$$\mathscr{D} = S_+ - S_w = \pi\rho \int_0^R \left\{\left(\frac{\partial\psi}{\partial x}\right)^2 - \psi\frac{\partial^2\psi}{\partial x^2}\right\}\frac{dr}{r}, \tag{4.11}$$

where ψ is the stream-function perturbation downstream. Components ψ_n given by (4.5) with real-exponential dependence on x evidently make no contribution to this integral, as would be expected since S_w must be the same for every cross-section downstream. After the substitution of (4.9) in (4.11), the orthogonality condition (4.10) shows that

$$\mathscr{D} = \pi\rho \sum_{n=1}^{m} \kappa_n^2 A_n^2. \tag{4.12}$$

Thus each wave component contributes separately to $\mathscr{D}$. Note that, according to these deductions, no drag could occur if the condition (4.8) were not satisfied.

The energy density of the downstream flow is

$$E = \pi\rho \int_0^R (u^2+v^2+w^2)\, r\, dr$$

$$= \pi\rho \int_0^R \left\{ \left(-U + \frac{1}{r}\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial x}\right)^2 + \left(-\Omega r + \frac{k\psi}{r}\right)^2 \right\} r\, dr$$

$$= E_0 + E_w, \quad \text{say,}$$

where E_0 is the energy density of the original, undisturbed flow and E_w is the part of E depending on ψ. Using the differential equation (4.1), and also the boundary conditions (4.4) in two integrations by parts, we find that

$$E_w = -2\pi\rho\Omega k \int_0^R \psi r\, dr + \pi\rho \int_0^R \left\{ \left(\frac{\partial\psi}{\partial x}\right)^2 + \psi\frac{\partial^2\psi}{\partial x^2} + k^2\psi^2 \right\} \frac{dr}{r}. \tag{4.13}$$

Next (4.9) is substituted, the orthogonality condition is again used, and the resulting expression is averaged with respect to x. The mean energy density of the lee-wave system is thus found to be

$$\bar{E}_w = \pi\rho k^2 \sum_{n=1}^{m} A_n^2. \tag{4.14}$$

The results so far are exact within the framework of the steady-flow model. But now time-dependent features are considered, and in order to warrant a simple approximate account of the energy balance the disturbances downstream are assumed to be small in amplitude. Accordingly, to estimate the distances over which the lee waves have become established at large but finite times, we may use the asymptotic first approximation given by linearized wave theory (e.g. see Lighthill 1967). That is, each component represented in (4.9) is taken as a uniform wave-train whose point of termination, say $x = x_n'$, has a constant velocity given by $dx_n'/dt = (c_g)_n - U < 0$, where $(c_g)_n$ is the forward group velocity. [Here we rely in effect on the familiar principle that the energy of a sinusoidal wave-train is transmitted at its group velocity.] As was noted earlier, linearized theory shows that the transients at the end of a developing wave-train cover a length $O(t^{\frac{1}{3}})$, which therefore eventually becomes an insignificant fraction of the total length of the train. Thus the present approximation is seen to be justified at sufficiently large times: the understanding is that time is large enough to discriminate distances that are small compared with the overall lengths of the wave-trains but large compared with the lengths of the transient zones and the wavelengths $2\pi/\kappa_n$, so that then the distributions of aggregate quantities such as energy are accurately represented by averages over these intermediate distances. The same idea was introduced in § 2.1 for the treatment of the water-wave problem, and it may be formally substantiated by use of the averaging operator defined in § 4.3 below.

Being stationary relative to the body, the established waves have a phase velocity U relative to the fluid. Hence the velocity $C_n = -dx_n'/dt = U - (c_g)_n$

at which the point of termination recedes downstream can be found by treating (4.6) as an expression for U in terms of the wave-number κ_n. That is, from

$$U = \frac{2\Omega R}{(\kappa_n^2 R^2 + j_n^2)^{\frac{1}{2}}},$$

we obtain

$$C_n = U - \frac{d(\kappa_n U)}{d\kappa_n} = \frac{U\kappa_n^2}{\kappa_n^2 R^2 + j_n^2} = \frac{U\kappa_n^2}{k^2}. \tag{4.15}$$

Since each wave component contributes separately to the expression (4.14) for E_w, it follows that the rate of increase of the total energy $\mathscr{E}$ in the wave system is approximately

$$\frac{d\mathscr{E}}{dt} = \pi\rho k^2 \sum_{n=1}^{m} C_n A_n^2 = \pi\rho U \sum_{n=1}^{m} \kappa_n^2 A_n^2. \tag{4.16}$$

Comparing (4.12) we see this to be the same as $\mathscr{D}U$, which was shown earlier (at the end of § 3.1) to be the rate of energy supply to the fluid at infinity. Thus, to a first approximation, the supplied energy appears to be manifested wholly in the lee-wave system.

4.3. *Reductio ad absurdum*

We come to the crucial point of this discussion on applying the impulse principle that was established in § 3. The fact that the flow near the body becomes steady as $t \to \infty$ implies that $\mathscr{D}$ becomes a constant, which is positive by hypothesis, and also that $(P_{\mathscr{S}})_x$ given by (3.14) becomes a constant. Hence the principle expressed by (3.7) and (3.13) means that $(P_V)_x \sim \mathscr{D}t$ as $t \to \infty$, where $(P_V)_x$ given by (3.15) is the component of impulse depending on vorticity perturbations throughout the fluid.

Thus it appears necessary that the lee-wave system should manifest a positive amount of impulse which, like the energy $\mathscr{E}$, grows without limit in proportion to the elongations of the component wave-trains. To clarify this idea, let $\partial(P_V)_x$ denote the density of impulse with respect to x, and suppose that the local mean value $\overline{\partial(P_V)_x}$ can be partitioned among the wave components, similarly to the way in which the mean energy density ($E_w \equiv \partial\mathscr{E}$) is expressed by (4.14). Then, in analogy with (4.16), one would expect that

$$\mathscr{D} = \sum_{n=1}^{m} [\overline{\partial(P_V)_x}]_n C_n,$$

where as before C_n denotes the rate of increase of the length over which the nth component is established.

The expression (3.15) for $(P_V)_x$ implies

$$\overline{\partial(P_V)_x} = \pi\rho \int_0^R \overline{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right)} r^2 dr = \pi\rho k^2 \int_0^R \overline{\psi} r\, dr, \tag{4.17}$$

by virtue of the differential equation (4.1) satisfied by ψ. But this mean value is precisely *zero*, because each component solution ψ_n of the differential equation is a sinusoidal function of x. Hence we conclude that the total impulse of the lee-wave system remains bounded as $t \to \infty$.

On the strength of this fact it becomes intuitively clear that the impulse principle is contradicted and therefore the assumption of steady upstream flow is incorrect. The analogy with the water-wave problem is impressive: we have again found that an assessment of the energy in the wave system reveals no inconsistency in the absence of upstream influence, but that an assessment of impulse is outstandingly discrepant. Some further discussion is needed, however, in order to establish the contradiction definitely.

First, it has to be confirmed that the steady flow behind the body cannot include disturbances independent of x (i.e. Taylor columns), which could make a positive contribution to (4.17). Considering the completeness of the sequence $\{\phi_n\}$, we see at once that the differential equation (4.1) with boundary conditions (4.4) generally has no solution independent of x; but an exception must be recognized in the case when exactly $kR = j_n$ for some $n > 1$, say $n = p$, so that $\psi_p = A_p \phi_p(r)$. The impossibility of an x-independent component of ψ in the extraordinary case may be argued by the fact that a contribution would be made by it to $\bar{E}_w$, from the first term on the right-hand side of (4.13), whereas none would be made to $\mathscr{D}$ as given by (4.11): hence the overall energy balance would not be satisfied. Alternatively we may consider that the angular-momentum perturbation downstream has a mean density

$$\overline{\partial N} = 2\pi\rho \int_0^R (\overline{w - |\Omega r|})\, r^2 dr = 2\pi\rho k \int_0^R \bar{\psi}\, r\, dr, \tag{4.18}$$

which is $2/k$ times the right-hand side of (4.17). It can easily be shown that

$$\int_0^R \phi_n(r)\, r\, dr = \frac{2^{\frac{1}{2}} R^2}{j_n}, \tag{4.19}$$

so that a component of ψ in the form $A_p\, \phi_p(r)$ with $A_p > 0$ would make a positive contribution to both (4.17) and (4.18). Also, as will be shown presently, such a disturbance would be uniform over virtually the whole distance to its extremity receding downstream. But the total perturbation N must always be zero, because the angular momentum of the whole fluid remains constant in the absence of an external torque and, by hypothesis, there is no effect upstream that can compensate a change of angular downstream of the body. Thus we require $A_p = 0$ to conform with $dN/dt = 0$ for all t.†

It remains to eliminate the possibility that a sufficient amount of impulse might somehow arise in transient effects at and between the terminations of the component wave-trains. In the light of the analogy with the water-wave problem this appears extremely unlikely, but to rule it out decisively we need to develop

† A referee has drawn attention to the possibility of a columnar wake in which the vorticity ζ is different from $k^2\psi/r$, and which therefore might manifest a positive impulse without violating the condition of angular-momentum conservation. This would, of course, be a real-fluid effect, of the kind that remains when the Reynolds number of the flow is extremely large, and it can reasonably be expected as a feature of rotating flow behind any sufficiently bluff body. Since our theoretical model does not allow this possibility, it seems prudent to limit present contentions to the case of slender streamlined bodies, such that there is no region of reversed flow downstream. On the other hand, bluff bodies appear intuitively to be even more liable to exert upstream influence!

an argument similar to Whitham's theory cited in §2. The starting point is a pair of exact equations for time-dependent axisymmetric flow: first the equation satisfied by the vorticity $\zeta = \partial v/\partial x - \partial u/\partial r$,

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial r} - \frac{1}{r}\left(v\zeta + 2w\frac{\partial w}{\partial x}\right) = 0, \tag{4.20}$$

which is derivable from the axial and radial equations of motion as given by Squire [1956, equations (10) and (11)]; and secondly the equation expressing conservation of azimuthal circulation

$$r\frac{\partial w}{\partial t} + ur\frac{\partial w}{\partial x} + v\frac{\partial (rw)}{\partial r} = 0, \tag{4.21}$$

which is equivalent to (12) in Squire's article. We note that the steady flow described by (4.9) is an exact solution of these equations, and also of their linearized forms. Hence, if the amplitudes A_n are considered to be of first-order smallness, say $O(\epsilon)$, the linearized equations may be used to obtain a consistent first approximation to each component wave-train in its entirity, including the time-dependent features at its downstream end. The main conclusions of the relevant theory have already been stated. Asymptotically for large t, each wave-train is uniform and steady over a length $C_n t$, and the transient zone at its downstream end has an effective length proportional to $t^{\frac{1}{2}}$. Furthermore, ζ and all other perturbations remain bounded in the transient zone, being of the same order of smallness there as in the steady part of the wave-train.

Now, we are concerned with the aggregate property $(P_V)_x$ that is required by the impulse principle to increase proportionally to t, and is the volume integral of the quantity $r\zeta$ which is uniformly bounded everywhere. Therefore, to obtain an asymptotic estimate of $(P_V)_x$, it is sufficient to use a form of equations (4.20) and (4.21) averaged over distances that are $o(t)$. A suitable averaging operator A is defined by

$$\bar{\zeta}(x,r,t) = \mathsf{A}\zeta = \frac{1}{2\xi}\int_{x-\xi}^{x+\xi} \zeta(X,r,t)\,dX, \tag{4.22}$$

in which $\xi = at^\beta$ with $\frac{1}{2} < \beta < 1$, so that $\xi = o(t)$. We then have

$$\mathsf{A}\left(\frac{\partial \zeta}{\partial x}\right) = \frac{\partial \bar{\zeta}}{\partial x},$$

$$\mathsf{A}\left(\frac{\partial \zeta}{\partial t}\right) = \frac{\partial \bar{\zeta}}{\partial t} + \frac{\beta\bar{\zeta}}{t} - \frac{\beta}{2t}\{\zeta(x+\xi,r,t) + \zeta(x-\xi,r,t)\}$$

$$\sim \frac{\partial \bar{\zeta}}{\partial t} \quad \text{for} \quad t\to\infty,$$

and similar results for the other terms in (4.20) and (4.21). Thus, taking non-linear terms to the right-hand side and writing $u' = u - |U|$, $w' = w - |\Omega r|$, we obtain from these equations the asymptotic approximations

$$\frac{\partial \bar{\zeta}}{\partial t} - U\frac{\partial \bar{\zeta}}{\partial x} + 2\Omega\frac{\partial \bar{w}'}{\partial x} = -\mathsf{A}\left[u'\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial r} - \frac{1}{r}\left(v\zeta + 2w'\frac{\partial w'}{\partial x}\right)\right], \tag{4.23}$$

$$\frac{\partial \bar{w}'}{\partial t} - U\frac{\partial \bar{w}'}{\partial x} - 2\Omega\bar{v} = -\mathsf{A}\left[u'\frac{\partial w'}{\partial x} + \frac{v}{r}\frac{\partial (rw')}{\partial r}\right]. \tag{4.24}$$

To obtain approximations to $O(\epsilon^2)$, the results of the linearized theory may be substituted on the right-hand side of (4.23) and (4.24). It has already been noted that, being sinusoidal in x, the steady parts of the component wave-trains make no contribution to mean values such as $\bar{\zeta}$ and $\bar{w}'$, and, by virtue of the specification $\beta > \frac{1}{2}$, all effects confined to the transient zones at the downstream ends of the wave-trains are evanescent under the averaging operation A. Everywhere in the lee-wave system except in the transient zones, the linearized theory gives as an asymptotic approximation

$$(u', v, w') = \frac{1}{r}\left(\frac{\partial\psi}{\partial r}, -\frac{\partial\psi}{\partial x}, k\psi\right), \quad \zeta = \frac{k^2\psi}{r},$$

which is the same as according to the exact steady-flow theory; and on substitution of these expressions the quantities within the brackets in (4.23) and (4.24) vanish identically. We also have asymptotically

$$\bar{v} = -\frac{1}{r}\frac{\partial\bar{\psi}}{\partial x}, \quad \bar{\zeta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{\psi}}{\partial r}\right).$$

Hence, upon eliminating $\bar{w}'$ between the reduced forms of (4.23) and (4.24), we obtain finally

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\bar{\psi}}{\partial r}\right) + \frac{4\Omega^2}{r}\frac{\partial^2\bar{\psi}}{\partial x^2} = 0. \tag{4.25}$$

It is worth further emphasis that to $O(\epsilon^2)$ this equation becomes exact asymptotically for large t, and so the approximation is certainly adequate for the present purpose.

The general solution of (4.25) subject to the boundary conditions

$$\bar{\psi}(x, 0, t) = 0, \quad \bar{\psi}(x, R, t) = 0, \tag{4.26}$$

is expressible in the form†

$$\bar{\psi} = \sum_{n=1}^{m} [f_n\{x + Ut + (c_0)_n t\} + g_n\{x + Ut - (c_0)_n t\}]\phi_n(r), \tag{4.27}$$

where

$$(c_0)_n = 2\Omega R/j_n = UkR/j_n. \tag{4.28}$$

In the representation of disturbances propagating downstream only, as required by the hypothesis of no upstream influence, all the f_n are relevant and the g_n are relevant for which n satisfies $j_n < kR$, so that $U - (c_0)_n > 0$. But it was shown earlier that at a fixed distance behind the body, say at $x = x_1$, every component of $\bar{\psi}$ must vanish asymptotically as the flow becomes steady. Hence, in using (4.27) to evaluate the expression for $d(P_V)_x/dt$ given by (3.15), we are faced with a set of results like

$$\int_{-\infty}^{x_1} \frac{\partial}{\partial t} f_n\{x + Ut + (c_0)_n t\}\,dx = \{U + (c_0)_n\} f_n\{x_1 + Ut + (c_0)_n t\}$$
$$\to 0 \quad \text{for} \quad t \to \infty,$$

† Note that long waves with isolated discontinuities in x and t can be justified as solutions of (4.25). The full linearized theory shows that such apparent discontinuities are in fact transient zones whose lengths are asymptotically proportional to $t^{\frac{1}{3}}$ (see Benjamin & Barnard 1964, §4). Hence the averaging operation A obliterates these zones even more quickly than the terminal zones of sinusoidal wave-trains.

from which we conclude that the rate of increase of the total impulse becomes zero asymptotically—in contradiction of the impulse principle if $\mathscr{D} \neq 0$.

Thus the hypothesis of no upstream influence, upon which the theoretical model defined in §4.1 is based, has been shown to be incompatible with the supposition that wave resistance occurs, and the following conclusion has been established by contradiction: *If, in a uniformly rotating inviscid fluid filling an infinitely long cylinder, a solid body is moved axially with a constant velocity* $U < 0{\cdot}522\Omega R$, *so that it experiences wave resistance, then a finite disturbance of the fluid* ALWAYS *occurs over a continually increasing distance ahead of the body.*

4.4. *Impulse of Taylor columns*

Unlike its precedent in the water-wave problem (§§2.3, 2.4), the argument just used cannot easily be extended to evaluate the columnar motions whereby upstream influence is propagated. In general many long-wave modes are entailed, not only one as before, and the relationships between time-dependent features in front and in the rear of the obstacle are rather complicated. Further uses of the impulse principle are left for subsequent study, and to complete the present discussion we merely confirm that the contradiction demonstrated in §4.3 is indeed resolved by allowing for upstream influence.

The linearized theory evoked in §§4.2 and 4.3 indicates that steady periodic wave-trains cannot be established ahead of the body, because for finite values of the axial wave-number κ_n the group velocity is directed downstream. But the difference between phase and group velocity vanishes in the limit $\kappa_n \to 0$ (just as for surface waves on water of finite depth), and thus the restraint on forward transmission does not apply to extremely long waves. The theory of such waves is largely recovered by what has been worked out in §4.3, and full accounts may be found in several papers (e.g. Trustrum 1964; Benjamin & Barnard 1964; Lighthill 1967). In the following description of the relevant set of fundamental solutions we continue to use the moving reference frame, in which the body appears stationary. We note that at moderate Rossby numbers (i.e. kR not extremely large) the amplitudes of the disturbances now in question are of second-order smallness, in the sense that the lee-wave amplitudes are of first-order smallness.

It is assumed that the forcing effect of the body has been acting steadily for a fairly long time. Then in the interior of a particular wave the motion is independent of x, specifically such that the stream-function perturbation is

$$\tilde{\psi}_n = const. \times \phi_n(r); \tag{4.29}$$

but the extremity of the wave propagates with either of the velocities $-U \pm (c_0)_n$, where $(c_0)_n$ is given by (4.28). Thus downstream propagation is possible for all n; but upstream propagation $[-U + (c_0)_n > 0]$ is possible only for $n \leqslant m$, where as before m is defined by the inequalities $j_{m+1} \geqslant kR > j_m$.

The present aspect can be nicely appreciated by imagining that the steady flow established immediately in front of the body has been measured, so that the stream-function perturbation at a certain x is a known function of r, say $F(r)$. In general this is expressible as the Fourier–Bessel series

$$F(r) = \sum_{n=1}^{\infty} B_n \phi_n(r), \tag{4.30}$$

in which

$$B_n = \int_0^R F(r)\, \phi_n(r) \frac{dr}{r},$$

by virtue of (4.10). We now recognize that the series has two parts with contrasting interpretations. The terms with $n > m$ must be contributed by component disturbances of the form (4.5) with real-exponential dependence on x, and so this contribution to the upstream flow would be found to diminish rapidly into insignificance with increasing distance from the body. On the other hand the terms with $n \leqslant m$ are accountable only to the long-wave modes (4.29) which, at large times, comprise a disturbance extending to great distances upstream. This is the kind of disturbance commonly known by the term Taylor column.

Let P_x^+ denote the impulse of the forward column. Its derivative with respect to time may be deduced from the first equation in (4.17), together with the facts expressed by (4.27) and (4.28). In this way, and by using the property (4.19) of the Sturm–Liouville functions ϕ_n, one obtains

$$\begin{aligned}\frac{dP_x^+}{dt} &= -\pi\rho \sum_{n=1}^{m} \{(c_0)_n - U\} B_n \int_0^R \frac{d}{dr}\left(\frac{1}{r}\frac{d\phi_n}{dr}\right) r^2 dr \\ &= \frac{\pi\rho U}{R^2} \sum_{n=1}^{m} j_n (kR - j_n) B_n \int_0^R \phi_n r\, dr \\ &= 2^{\frac{1}{2}}\pi\rho U \sum_{n=1}^{m} (kR - j_n) B_n. \end{aligned} \tag{4.31}$$

It readily appears that this expression is positive for reasonable models of the column. [For instance, if $-u \geqslant -U$ at $r = R$ everywhere upstream, as would be expected, then none of the B_n's can be negative.] We cannot equate this expression to the drag $\mathscr{D}$, however, because columnar disturbances on the downstream side generally also contribute to the total impulse.

I am indebted to Mr L. E. Fraenkel for valuable advice about the presentation of this paper.

Appendix A. Reduction of a volume integral

The integral is

$$\mathbf{I} = \int_V \mathbf{x} \times (\nabla \times \mathbf{Q})\, d\tau, \tag{A 1}$$

where V is a volume with internal boundary $\mathscr{S}$ and external boundary Σ, $\mathbf{x}$ is the three-dimensional position vector, and $\mathbf{Q}(\mathbf{x})$ is a differentiable vector field. Considering the vector identity

$$\mathbf{x} \times (\nabla \times \mathbf{Q}) = -\mathbf{Q} \times (\nabla \times \mathbf{x}) + \nabla(\mathbf{x} \cdot \mathbf{Q}) - (\mathbf{x} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{x},$$

we note that the first term on the right-hand side is zero (since $\mathbf{x} = \frac{1}{2}\nabla x_i^2$), and the last term reduces to $-\mathbf{Q}$. Furthermore,

$$(\mathbf{x}\cdot\nabla)\,\mathbf{Q} = \frac{\partial(x_i\mathbf{Q})}{\partial x_i} - 3\mathbf{Q}.$$

Hence (A 1) is equivalent to

$$\begin{aligned}\mathbf{I} &= \int_V \left\{2\mathbf{Q} + \nabla(\mathbf{x}\cdot\mathbf{Q}) - \frac{\partial(x_i\mathbf{Q})}{\partial x_i}\right\} d\tau \\ &= 2\int_V \mathbf{Q}\,d\tau + \int_{\mathscr{S}+\Sigma} \{(\mathbf{x}\cdot\mathbf{Q})\,d\mathbf{s} - \mathbf{Q}(\mathbf{x}\cdot d\mathbf{s})\} \\ &= 2\int_V \mathbf{Q}\,d\tau - \int_{\mathscr{S}+\Sigma} \mathbf{x}\times(\mathbf{Q}\times d\mathbf{s}), \end{aligned} \quad \text{(A 2)}$$

which is the formula used in §3.2 and appendix C.

Appendix B. Reduction of a surface integral

The integral is

$$\mathbf{J} = \int_{\mathscr{S}} \mathbf{x}\times(\nabla H\times d\mathbf{s}), \quad \text{(B 1)}$$

where $\mathscr{S}$ is a simple closed surface and $H(\mathbf{x})$ is a differentiable scalar function of the three-dimensional position vector $\mathbf{x}$. The principles used in the following argument are standard (e.g. see Korn & Korn 1961, chapter 5).

We consider that $\mathscr{S}$ has the representation

$$\mathbf{x} = \mathbf{r}(\alpha,\beta),$$

in which α, β are scalar parameters, and without loss of generality we may assume that $\alpha, \beta \in [0, 1]$. The vector element of area is expressible as

$$d\mathbf{s} = \left(\frac{\partial\mathbf{r}}{\partial\alpha}\times\frac{\partial\mathbf{r}}{\partial\beta}\right) d\alpha\, d\beta,$$

so that

$$\begin{aligned}\nabla H\times d\mathbf{s} &= \left\{\left(\nabla H\cdot\frac{\partial\mathbf{r}}{\partial\beta}\right)\frac{\partial\mathbf{r}}{\partial\alpha} - \left(\nabla H\cdot\frac{\partial\mathbf{r}}{\partial\alpha}\right)\frac{\partial\mathbf{r}}{\partial\beta}\right\} d\alpha\, d\beta \\ &= \left\{\left(\frac{\partial H(\mathbf{r})}{\partial\beta}\right)\frac{\partial\mathbf{r}}{\partial\alpha} - \left(\frac{\partial H(\mathbf{r})}{\partial\alpha}\right)\frac{\partial\mathbf{r}}{\partial\beta}\right\} d\alpha\, d\beta.\end{aligned}$$

Hence (B 1) is equivalent to

$$\mathbf{J} = \int_0^1\int_0^1 \left\{\left(\mathbf{r}\times\frac{\partial\mathbf{r}}{\partial\alpha}\right)\frac{\partial H(\mathbf{r})}{\partial\beta} - \left(\mathbf{r}\times\frac{\partial\mathbf{r}}{\beta\partial}\right)\frac{\partial H(\mathbf{r})}{\partial\alpha}\right\} d\alpha\, d\beta. \quad \text{(B 2)}$$

Since $\mathscr{S}$ is a closed surface, $H(\mathbf{r})$ is a periodic function of α and β. Accordingly, if the two components of (B 2) are integrated by parts over the period of β and α respectively, the integrated terms cancel and there follows

$$\begin{aligned}\mathbf{J} &= \int_0^1\int_0^1 H(\mathbf{r})\left\{-\frac{\partial}{\partial\beta}\left(\mathbf{r}\times\frac{\partial\mathbf{r}}{\partial\alpha}\right) + \frac{\partial}{\partial\alpha}\left(\mathbf{r}\times\frac{\partial\mathbf{r}}{\partial\beta}\right)\right\} d\alpha\, d\beta \\ &= 2\int_0^1\int_0^1 H(\mathbf{r})\left(\frac{\partial\mathbf{r}}{\partial\alpha}\times\frac{\partial\mathbf{r}}{\partial\beta}\right) d\alpha\, d\beta \\ &\equiv 2\int_{\mathscr{S}} H\, d\mathbf{s}, \end{aligned} \quad \text{(B 3)}$$

which is the formula used in §3.2.

Appendix C. Alternative derivation of impulse principle

The derivation in §3.2 involved a reduction of integrals expressing $d\mathbf{P}/dt$. Here a reduction of the original expression (3.7) for $\mathbf{P}$ is considered, which reveals its equivalence with Kelvin's definition of fluid impulse. Incorporating a generalization that was not needed in the discussion of upstream influence, we take a fixed frame of reference (so that the fluid at infinity has no axial motion) and let the axial velocity of the body be an arbitrary function of time.

Recapitulating (3.7) we have

$$\frac{\mathbf{P}}{\frac{1}{2}\rho} = \int_V \mathbf{x}\times\boldsymbol{\omega}\,d\tau + \int_{\mathscr{S}} \mathbf{x}\times(\mathbf{u}\times d\mathbf{s}), \tag{C1}$$

in which $\boldsymbol{\omega} = \nabla\times\mathbf{u}$. Applied to the volume integral in (C1), the formula established in appendix A gives

$$\int_V \mathbf{x}\times(\nabla\times\mathbf{u})\,d\tau = 2\int_V \mathbf{u}\,d\tau - \int_{\mathscr{S}+\Sigma} \mathbf{x}\times(\mathbf{u}\times d\mathbf{s}).$$

Thus, the surface integral in (C1) is cancelled and we are left with

$$\begin{aligned}\mathbf{P} &= \rho\int_V \mathbf{u}\,d\tau - \tfrac{1}{2}\rho\int_\Sigma \mathbf{x}\times(\mathbf{u}\times d\mathbf{s})\\ &= \mathbf{m}+\mathbf{P}_\Sigma, \quad \text{say.}\end{aligned} \tag{C2}$$

Here $\mathbf{m}$ is the momentum of the fluid and $\mathbf{P}_\Sigma$ is an effect developed over the outer boundary Σ, which is comprised from the rigid cylindrical surface $r = R$ and cross-sections at $\pm\infty$. To confirm the interpretation of $\mathbf{P}$ explained in §3.3, which corresponds to Kelvin's definition, it remains to show that $\mathbf{P}_\Sigma$ represents the time-integral of the reaction across Σ.

[Note incidentally that if the flow were *irrotational* (i.e. if $W \equiv 0$) and therefore a velocity potential Φ existed such that $\mathbf{u} = \nabla\Phi$, then the formula established in appendix B would give immediately

$$\mathbf{P}_\Sigma = -\rho\int_\Sigma \Phi\,d\mathbf{s}. \tag{C3}$$

This is a familiar representation of the property just described, in the case of irrotational flow (cf. Kochin, Kibel & Roze 1964, §7.7). After the substitution $\mathbf{u} = \nabla\Phi$, the volume integral $\mathbf{m}$ is reducible to two surface integrals, one of which cancels $\mathbf{P}_\Sigma$ as given by (C3). The result is

$$\mathbf{P} = -\rho\int_{\mathscr{S}} \Phi\,d\mathbf{s}, \tag{C4}$$

which is a well-known expression for the Kelvin impulse (i.e. the added mass for unit velocity of translation) generated by a body moving along a straight path (cf. Lamb 1932, §121).]

In general, recalling that u denotes the axial component of $\mathbf{u}$, we have

$$m_x = \rho \int_V u\,d\tau, \tag{C5}$$

$$(P_\Sigma)_x = -\tfrac{1}{2}\rho R^2 \int_0^{2\pi}\int_{-\infty}^{\infty} u(x, R, \theta)\,dx\,d\theta. \tag{C6}$$

But the axial component of the equation of motion (3.5) reduces on the surface $r = R$ to

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left\{\frac{p}{\rho} + \tfrac{1}{2}|\mathbf{u}|^2\right\};$$

and the radial component shows that $p/\rho + \frac{1}{2}|\mathbf{u}|^2 \to p'$ (*const.*) for $x \to \pm\infty$. Hence (C6) gives

$$\frac{d}{dt}(P_\Sigma)_x = \pi R^2 (p'_{x=\infty} - p'_{x=-\infty}), \tag{C7}$$

which is the anticipated result.

Thus it appears that

$$\frac{dP_x}{dt} = \frac{dm_x}{dt} + \pi R^2 (p'_{x=\infty} - p'_{x=-\infty}). \tag{C8}$$

This constitutes a statement of the impulse principle, for the right-hand side evidently equals the applied force $\mathscr{D}$.

Finally, a simple expression for the momentum m_x will be derived. Again recalling that x denotes the axial co-ordinate and u the axial component of $\mathbf{u}$, we note that

$$\nabla\cdot(x\mathbf{u}) = x\nabla\cdot\mathbf{u} + (\nabla x)\cdot\mathbf{u} = (\nabla x)\cdot\mathbf{u} = u.$$

Hence (C5) leads to $\quad m_x = \int_V \nabla\cdot(x\mathbf{u})\,d\tau = \int_{\mathscr{S}+\Sigma} x\mathbf{u}\cdot d\mathbf{s}.$

The surface integral over Σ is zero, and that over $\mathscr{S}$ may be reduced by use of the kinematical boundary condition

$$\mathbf{u}\cdot d\mathbf{s} = U(ds)_x \quad \text{on} \quad \mathscr{S}.$$

Thus we deduce

$$m_x = \int_{\mathscr{S}} x\mathbf{u}\cdot d\mathbf{s} = U\int_{\mathscr{S}} x(ds)_x = -U \times (\textit{volume of body}), \tag{C9}$$

which is the result noted in §3.1.

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